Outer product:

Let A^{ij}_{k} and B^{p}_{q} are two tensors. and A^{ij}_{k} has total rank three and, therefore, has N^3 components. B^{p}_{q} has total rank two and N^2 components.

- Each component of one tensor is multiplied by every component of the other. The resulting set of quantities gives a tensor whose rank is the sum of the ranks of the two original tensors.

We write the transformation equations for A^{ij}_{k} and B^{p}_{q} as

$$\bar{A}^{\alpha\beta}_{\gamma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} A^{ij}_{k}, \quad \text{—— (9)}$$

$$\bar{B}^{\rho}_{\sigma} = \frac{\partial \bar{x}^{\rho}}{\partial x^{p}} \frac{\partial x^{q}}{\partial \bar{x}^{\sigma}} B^{p}_{q} \quad \text{—— (10)}$$

Now

$$\bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\rho}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial \bar{x}^{\rho}}{\partial x^{p}} \frac{\partial x^{q}}{\partial \bar{x}^{\sigma}} A^{ij}_{k} B^{p}_{q} \quad \text{—— (11)}$$

Let us define $C^{ijp}_{kq} = A^{ij}_{k} B^{p}_{q}$, $\bar{C}^{\alpha\beta\rho}_{\gamma\sigma} = \bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\rho}_{\sigma}$. — (12)

- Next, we can write Eq. (11) as

$$\bar{C}^{\alpha\beta\rho}_{\gamma\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial \bar{x}^{\rho}}{\partial x^{p}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial x^{q}}{\partial \bar{x}^{\sigma}} C^{ijp}_{kq} \quad \text{—— (13)}$$

tensor of contravariant rank 3 and covariant rank 2;

C^{ijp}_{kq} — has total rank 5. and hence N^5 components. each of which is the product of one component A^{ij}_{k} with one B^{p}_{q}.

Eq (13) defines the outer product or Kronecker product of two tensors. This can also be extended to more than two tensors.

Inner product of two tensors: –

Let A^{ij}_{k} and B^{l}_{m} be two tensors. Consider the set of functions $A^{ij}_{k} B^{k}_{m}$ with i, j and m free indices and the index k is summed over from $k = 1, 2, \dots N$.

There are three free indices in the function $A^{ij}_{k} B^{k}_{m}$, therefore, the number of such functions will be N^3.

In the previous lecture note we have seen than

$$\bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\rho}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial \bar{x}^{\rho}}{\partial x^{l}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ij}_{k} B^{l}_{m} \quad \text{—} (1)$$

for $\rho = \gamma$, in above expression and summing over γ.

$$\bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\gamma}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{l}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ij}_{k} B^{l}_{m} \quad \text{—} (2)$$

Since $\frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{l}} = \delta^{k}_{l}$, thus we have

$$\bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\gamma}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} \delta^{k}_{l} A^{ij}_{k} B^{l}_{m}$$

Using the property of delta function, we can write

$$\bar{A}^{\alpha\beta}_{\gamma}\,\bar{B}^{\gamma}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial \bar{x}^{\beta}}{\partial x^{j}}\frac{\partial x^{m}}{\partial \bar{x}^{\sigma}}\,A^{ij}_{k}B^{k}_{m} \qquad (3)$$

Eq. (3) shows that $A^{ij}_{k}B^{k}_{m}$ transform like the component of contravariant rank 2 and covariant rank 1. Let us define

$$\bar{C}^{\alpha\beta}_{\sigma} = \bar{A}^{\alpha\beta}_{\gamma}\,\bar{B}^{\gamma}_{\sigma} \qquad (4)$$

$$\text{and } C^{ij}_{m} = A^{ij}_{k}B^{k}_{m} \qquad (5)$$

In eq (5) C^{ij}_{m} is called as the inner product of two tensors $A^{ij}_{k}\,B^{k}_{m}$.

<u>H.W.</u> If X^{ij}_{k} and Y^{l}_{m} are two tensors, Show that $X^{ij}_{k}Y^{i}_{m}$ is not a tensor.

Contraction of a tensor:

Let A^{ijk}_{lm} be a tensor, R. rank 5, total N^5 component. Contravariant rank 3, Covariant rank 2,

for $l = i$, A^{ijk}_{im} will have N^3 components since index i will be summed over.

Let us write transformation equation of A^{ijk}_{lm}

$$\bar{A}^{\alpha\beta\gamma}_{\rho\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial \bar{x}^{\beta}}{\partial x^{j}}\frac{\partial \bar{x}^{\gamma}}{\partial x^{k}}\frac{\partial x^{l}}{\partial \bar{x}^{\rho}}\frac{\partial x^{m}}{\partial \bar{x}^{\sigma}}A^{ijk}_{lm}$$

Now taking $\rho = \alpha$, and summing over α

$\cancel{\mathcal{B}}$.

$$\bar{A}^{\alpha\beta\gamma}_{\alpha\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{k}} \frac{\partial x^{l}}{\partial \bar{x}^{\alpha}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ijk}_{lm}$$

$$= \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{k}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} \delta^{l}_{i} A^{ijk}_{lm}$$

or $$\bar{A}^{\alpha\beta\gamma}_{\alpha\sigma} = \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{k}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ijk}_{im}$$

$A^{ijk}_{im} \rightarrow$ Contravariant rank 2
Covariant rank 1

When a tensor is contracted by making one of its covariant index equal to its contravariant index, then ~~the~~ resultant quantity is a tensor whose covariant and contravariant indices are reduced by one and therefore the total rank is reduced by two. This process is called the contraction of a tensor.